Semiclassical spectrum for BMN string in $Sch_5 \times S^5$

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Abstract

We investigate the algebraic curve for string in $Sch_5 \times S^5$. We compute the semiclassical spectrum for BMN string in $Sch_5 \times S^5$ from the algebraic curve. We compare our results with the anomalous dimensions in $sl(2)$ sector of the null dipole deformation of $\mathcal{N} = 4$ super Yang-Mills theory.

1 Introduction

Spectrum of superstrings in $AdS_5 \times S^5$ is related to the spectrum of scaling dimensions in planar $\mathcal{N} = 4$ supersymmetric Yang-Mills theory via the AdS/CFT duality [1–3]. Integrability on both sides of the duality helps us dramatically finding and understanding the AdS/CFT spectrum (For a big review, see [4]). In the $\mathcal{N} = 4$ supersymmetric Yang-Mills theory, the planar anomalous dimension matrix of infinitely long composite operators corresponds to Hamiltonian of integrable spin chain [5]. This implies that the spectrum can be solved efficiently by the Bethe ansatz.

On the string side, classical integrability of superstrings in $AdS_5 \times S^5$ follows from the existence of an infinite number of conserved charges [6] generated by the monodromy matrix of the Lax connection. Algebraic curve for classical solution of superstring in $AdS_5 \times S^5$ [7–11] can be obtained from the Lax connection. It plays an important role in studying the semiclassical strings in $AdS_5 \times S^5$.

In recent years, much attention has been enjoyed by the integrable deformations of AdS/CFT. One intriguing example is the Schrödinger spacetime [12–14]. Schrödinger spacetime can be obtained from AdS background by an appropriate TsT (T-duality-shift-T-duality) transformation [15] or null Melvin twist and has been shown to be classically integrable [16–18]. String theories in Schrödinger spacetime is dual to null dipole deformed field theories [19] (see also [20–22]). It is interesting to study the spectrum on both sides of the Schrödinger/dipole CFT duality with the methods of integrability.

Integrability in null dipole deformed $\mathcal{N} = 4$ super Yang-Mills was discussed in detail in [23]. The dipole deformation can be described as a Jordan cell Drinfeld-Reshetikhin twist [24, 25] in the spin chain picture. The traditional Bethe ansatz is inapplicable due to the absence of a vacuum state. One-loop spectrum of the nontrivial twisted $sl(2)$ sector was instead obtained from the Baxter equation. In the large J limit, the anomalous dimension of the ground state perfectly matches the classical energy of the BMN string at order J^{-1}.

The purpose of this paper is to study the Schrödinger/dipole CFT duality by comparing semi-classical spectrum around classical string solutions to anomalous dimension of operators in the $sl(2)$ sector at order J^{-2} in the large J limit. One reason to study the order J^{-2} terms is that in the well studied AdS_5/CFT_4 correspondence, the gauge theory and string results match at order J^{-2} in the

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BMN limit [26]. One can expect that the null dipole deformation preserves this matching. Another reason is that at order J^{-2} we should consider one-loop quantum string theory corrections to the string energy, while the previous test at order J^{-1} involve purely bosonic classical string energies. We compute fluctuation energies of the excitations and the one-loop shift of the ground sate energy from algebraic curve. We show that semiclassical spectrum around the BMN string solution perfectly matches the spin chain prediction.

This paper is organized as follows. In section 2 we discuss the $Sch_5 \times S^5$ background and TsT transformation in detail. We discuss the algebraic curve for strings in this background and obtain the quasi-momenta for the BMN string. In section 3, we review the algebraic curve method for computing the fluctuation energies around classical string solutions. Then we compute the semiclassical spectrum for the BMN strings. In section 4, we compare string theory results obtained in section 3 with the 1-loop spectrum in the $sl(2)$ sector of the null dipole deformation of $\mathcal{N} = 4$ super Yang-Mills theory.

2 Algebraic curve for strings in $Sch_5 \times S^5$

2.1 $Sch_5 \times S^5$ from TsT transformation

Schrödinger spacetime can be constructed by applying a TsT transformations to the AdS background [12–14,19]. In this paper we are interested in a particular case of $Sch_5 \times S^5$ obtained by acting a TsT transformation on $AdS_5 \times S^5$ [1]. We begin with the $AdS_5 \times S^5$ solution of type IIB supergravity

$$\begin{aligned} ds^2 &= R^2\left(\frac{-2dx^+dx^- + (dx^1)^2 + (dx^2)^2 + dz^2}{z^2} + (d\psi + A)^2 + ds^2_{\mathbb{CP}^2}\right), && (2.1)\\ ds^2_{\mathbb{CP}^2} &= d\phi_1^2 + \frac{1}{4}\sin^2\phi_1^2(\cos\phi_1^2(d\theta_1 + \cos\phi_2 d\theta_2)^2 + d\phi_2^2 + \sin^2\phi_2 d\theta_2^2), && (2.2)\\ A &= \frac{1}{2}\sin^2\phi_1(d\theta_1 + \cos\phi_2 d\theta_2). && (2.3)\end{aligned}$$

The five-form field strength is given by

$$F_5 = 4R^4(-\frac{1}{z^5}dx^+ \wedge dx^- \wedge dx^1 \wedge dx^2 \wedge dz + \mathrm{vol}(S^5)). \quad (2.4)$$

We perform a TsT transformation to this geometry. We make a first T-duality along ψ, followed a shift $x^- \to x^- - \mu\psi$, and then apply a second T-duality along ψ coordinate. After this TsT transformation, the solution reads

$$\begin{aligned} ds^2 &= R^2\left(-\mu^2\frac{(dx^+)^2}{z^4} + \frac{-2dx^+d\tilde{x}^- + (dx^1)^2 + (dx^2)^2 + dz^2}{z^2} + (d\tilde{\psi} + A)^2 + ds^2_{\mathbb{CP}^2}\right), && (2.5)\\ B_2 &= \mu R^2(d\tilde{\psi} + A) \wedge \frac{dx^+}{z^2}. && (2.6)\end{aligned}$$

The five-form F_5 is invariant under the transformation. The TsT translation preserves the symmetries that commute with $\hat{J}$ and $\hat{P}_-$. The symmetries of $SO(4,2)$ that commute with $\hat{P}_-$ generate the Schrödinger group. The Schrödinger group contains Galilean group as a subgroup and has two more generators corresponding to a non-relativistic scale transformation $\hat{D} + \hat{M}_{+-}$ and a special conformal transformation $\hat{K}_-$. The energy of the string is defined as the global charge associated with the symmetry $(\hat{P}_+ + \hat{K}_-)/\sqrt{2}$ which is related to the non-relativistic scale transformation $\hat{D} + \hat{M}_{+-}$ by a similarity transformation. Holography enable one to compute the non-relativistic conformal dimensions of operators at strong coupling as the energies of strings.

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[1] A more general class of Schrödinger deformations of $AdS_5 \times X_5$ has been studied in [27].

The relations between the original and dual coordinates are

$$d\psi = d\tilde{\psi} + *\mu\frac{dx^+}{z^2}, \tag{2.7}$$

$$dx^- = d\tilde{x}^- + *\mu(d\tilde{\psi} + *\mu\frac{dx^+}{z^2} + A). \tag{2.8}$$

We consider the closed strings on the deformed background. The dual coordinates satisfy periodic boundary conditions. The original coordinates have the following twisted boundary conditions

$$x^-(2\pi) - x^-(0) = LJ, \tag{2.9}$$

$$\psi(2\pi) - \psi(0) = 2\pi m_1 - LP_-, \quad m_1 \in \mathbb{Z}, \tag{2.10}$$

where $L = 2\pi\mu/\sqrt{\lambda}$ is the deformation parameter in the dual field theory and

$$J = \frac{\sqrt{\lambda}}{2\pi}\int_0^{2\pi} d\sigma(\partial_\tau\psi + A_\tau), \tag{2.11}$$

$$P_- = \frac{\sqrt{\lambda}}{2\pi}\int_0^{2\pi} d\sigma\frac{-\partial_\tau x^+}{z^2}, \tag{2.12}$$

are global charges associated with symmetries $\hat{J}$ and $\hat{P}_-$, and $\sqrt{\lambda} = R^2/\alpha'$ is the square root of 't Hooft coupling.

2.2 Flat connection

Integrability of the string sigma model is preserved by TsT transformation, so strings in Schrödinger spacetime is integrable [16–18] (see also [28–42]).

We now construct the Lax connection for IIB superstring in $Sch_5 \times S^5$. The type IIB superstring in $AdS_5 \times S^5$ can be described by the a sigma-model in supercoset space of the super group $SU(2,2|4)$ over $SO(4,1) \times SO(5)$ [43]. To describe strings in Poincaré coordinates, we choose the coset representative as

$$g(x^\mu, z, \psi, \theta_i, \phi_i, \Theta) = B(x^\mu, z, \psi, \theta_i, \phi_i)e^{F(\Theta)}, \tag{2.13}$$

with

$$B(x^\mu, z, \psi, \theta_i, \phi_i) = e^{ix^+\hat{P}_+ + ix^-\hat{P}_- + ix^1\hat{P}_1 + ix^2\hat{P}_2}e^{-i\log(z)\hat{D}}e^{i\psi\hat{J}}B_1(\theta_i, \phi_i), \tag{2.14}$$

and Θ represents the fermionic coordinates. We use a matrix representation such that

$$\hat{P}_- = \left(\begin{array}{cccc|c} 0 & 0 & 0 & \sqrt{2} & \\ 0 & 0 & 0 & 0 & 0_{4\times4} \\ 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & \\ \hline & & 0_{4\times4} & & 0_{4\times4} \end{array}\right), \quad \hat{P}_+ = \left(\begin{array}{cccc|c} 0 & 0 & 0 & 0 & \\ 0 & 0 & \sqrt{2} & 0 & 0_{4\times4} \\ 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & \\ \hline & & 0_{4\times4} & & 0_{4\times4} \end{array}\right), \tag{2.15}$$

$$\hat{D} = \frac{i}{2}\mathrm{diag}(1,1,-1,-1|0,0,0,0), \quad \hat{J} = \frac{1}{2}\mathrm{diag}(1,1,1,-3|0,0,0,0).$$

The current associated with g can be defined as

$$\mathcal{J} = -g^{-1}dg = e^F\mathcal{J}_B e^{-F} - e^F de^{-F} = \mathcal{J}_B + \mathcal{J}_F, \tag{2.16}$$

where

$$\mathcal{J}_B = -B^{-1}dB = -\frac{i}{z}dx^\mu\hat{P}_\mu + \frac{i}{z}dz\hat{D} - id\psi B_1^{-1}\hat{J}B_1 - B_1^{-1}dB_1 \tag{2.17}$$

$$\mathcal{J}_F = \frac{\sinh\mathcal{M}}{\mathcal{M}}\nabla F + 2[F, \frac{\sinh^2\mathcal{M}/2}{\mathcal{M}^2}\nabla F], \tag{2.18}$$

with

$$\nabla \cdot := d \cdot + [\mathcal{J}_B, \cdot], \quad \mathcal{M}^2 \cdot := [F, [F, \cdot]]. \tag{2.19}$$

The Lie superalgebra $su(2,2|4)$ has a $\mathbb{Z}_4$ grading structure associated with a $\mathbb{Z}_4$ automorphism Ω. The automorphism Ω in this matrix representation is defined by

$$\Omega(M) = -KM^{st}K^{-1}, \quad K = \left(\begin{array}{cccc|cccc} 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right) \tag{2.20}$$

We can decompose the current $\mathcal{J}$ into

$$\mathcal{J} = \mathcal{J}^{(0)} + \mathcal{J}^{(1)} + \mathcal{J}^{(2)} + \mathcal{J}^{(3)}, \tag{2.21}$$

where $\Omega(\mathcal{J}^{(i)}) = i^n \mathcal{J}$. Then the equation of motion of string in $AdS_5 \times S^5$ is equivalent to the conservation of the Noether current

$$d * k = 0, \quad k = g(\mathcal{J}^{(2)} - \frac{1}{2}\mathcal{J}^{(1)} + \frac{1}{2}\mathcal{J}^{(3)})g^{-1}. \tag{2.22}$$

Lax connection $L(x)$ for the superstring in $AdS_5 \times S^5$ has been derived in [6]

$$\mathcal{L} = \mathcal{J}^{(0)} + \frac{x^2+1}{x^2-1}\mathcal{J}^{(2)} - \frac{2x}{(x^2-1)} * \mathcal{J}^{(2)} + \sqrt{\frac{x+1}{x-1}}\mathcal{J}^{(1)} + \sqrt{\frac{x-1}{x+1}}\mathcal{J}^{(3)}. \tag{2.23}$$

If the current satisfies the equation of motion, the Lax connection is flat

$$d\mathcal{L} - \mathcal{L} \wedge \mathcal{L} = 0. \tag{2.24}$$

Using this flat connection, one can define the monodromy matrix

$$T(x) = \mathrm{P}\exp(\int_0^{2\pi} d\sigma \mathcal{L}_\sigma(x)). \tag{2.25}$$

The eigenvalues of $T(x)$ do not depend on τ and generate an infinite number of conserved quantities. The current components $\mathcal{J}_\alpha$ do not have an explicit dependence on x^- and ψ. Then the Lax connection $\mathcal{L}$ in the undeformed case can be used to derive a Lax connection and thus quasi-momenta for strings in $Sch_5 \times S^5$ background.

The quasi-momenta $p_i(x)$ are functions defined from the eigenvalues $\{e^{\hat{p}_i}|e^{\tilde{p}_i}\}$ of the monodromy matrix $T(x)$. They are generating functions of conserved physical quantities. For instance, we can read the conserved global charges from the behavior at large x. Large x asymptotic properties of the quasi-momenta for strings with twisted boundary condition are more complex than those for close strings. Below we analysis the asymptotic behavior of the quasi-momenta.

At $x \to \infty$, the expansion of the Lax connection is

$$\mathcal{L} = -g^{-1}dg - \frac{2}{x}g^{-1} * kg + O(\frac{1}{x^2}). \tag{2.26}$$

Expanding the monodromy matrix, we get

$$\begin{aligned} \tilde{T} \equiv & g(\tau,0)Tg(\tau,0)^{-1} \\ = & g(\tau,0)g(\tau,2\pi)^{-1}\left(1 - \frac{2}{x}\int_0^{2\pi} d\sigma k_\tau + O(\frac{1}{x^2})\right) \\ = & e^{iL(P_-\hat{J} - J\hat{P}_-)}\left(1 - \frac{2}{x}\int_0^{2\pi} d\sigma k_\tau + O(\frac{1}{x^2})\right) \end{aligned} \tag{2.27}$$

In our representation, it takes the form

$$\tilde{T} = \left(\begin{array}{cccc|c} 0 & 0 & 0 & -i\sqrt{2}LJ & \\ 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0_{4\times 4} \\ 0 & 0 & 0 & 0 & \\ \hline & & 0_{4\times 4} & & \exp(\frac{i}{2}LP_-\mathrm{diag}\,(1,1,1,-3)) \end{array}\right) + \frac{1}{\sqrt{\lambda}x}Q + O(\frac{1}{x^2}), \tag{2.28}$$

where Q satisfies $Q_{\hat{4}\hat{1}} = -2\pi i\sqrt{2}P_-$. Behavior of the quasi-momenta at $x \to \infty$ reads

$$\left(\begin{array}{c} \hat{p}_1 \\ \hat{p}_2 \\ \hat{p}_3 \\ \hat{p}_4 \\ \hline \tilde{p}_1 \\ \tilde{p}_2 \\ \tilde{p}_3 \\ \tilde{p}_4 \end{array}\right) = \left(\begin{array}{c} 2\sqrt{\frac{\pi LJP_-}{\sqrt{\lambda}x}} \\ 0 \\ 0 \\ -2\sqrt{\frac{\pi LJP_-}{\sqrt{\lambda}x}} \\ \hline \frac{1}{2}LP_- \\ \frac{1}{2}LP_- \\ \frac{1}{2}LP_- \\ -\frac{3}{2}LP_- \end{array}\right) + \frac{2\pi}{\sqrt{\lambda}x}\left(\begin{array}{c} O(1) \\ \Delta+S \\ -\Delta+S \\ O(1) \\ \hline +J_1+J_2-J_3 \\ +J_1-J_2+J_3 \\ -J_1+J_2+J_3 \\ -J_1-J_2-J_3 \end{array}\right) + O(x^{-2}). \tag{2.29}$$

The quasi-momenta $\hat{p}_1$ and $\hat{p}_4$ are connected by a square root cut which has a branch point at infinity. The quasi-momenta satisfy

$$\oint_\infty dx(\tilde{p}_1^2 + \tilde{p}_2^2 + \tilde{p}_3^2 + \tilde{p}_4^2 - \hat{p}_1^2 - \hat{p}_2^2 - \hat{p}_3^2 - \hat{p}_4^2) = 0, \tag{2.30}$$

so we still have the constraint between length and filling fractions (see [9] [11]).

2.3 BMN string

We now exemplify the discussion above with BMN string solution in $Sch_5 \times S^5$ presented in [23]

$$\begin{aligned} &\phi_i = \theta_i = x^2 = x^1 = 0, \quad \psi = \mu m\sigma + \omega\tau, \\ &x^+ = \frac{\tan(\kappa\tau)}{\sqrt{2}}, \quad x^- = \mu\omega\sigma + \frac{\kappa\tan(\kappa\tau)}{2m}, \\ &z = \sqrt{\frac{\kappa}{\sqrt{2}m}}\sec(\kappa\tau). \end{aligned} \tag{2.31}$$

Virasoro constraint gives

$$\mu^2 m^2 - \kappa^2 + \omega^2 = 0. \tag{2.32}$$

The conserved global charges are

$$\Delta = \sqrt{\lambda}\kappa, \quad P_- \equiv -M = -\sqrt{\lambda}m, \quad J = \sqrt{\lambda}\omega, \tag{2.33}$$

where we denote Δ as the global energy of the string. Then the classical energy of the BMN string is given by

$$\Delta = \sqrt{J^2 + \mu^2 M^2}. \tag{2.34}$$

The quasi-momenta of the BMN string are

$$\begin{aligned}
\hat{p}_1 &= -\hat{p}_4 = \frac{2\pi\kappa x\sqrt{1-x\sin 2\gamma}}{x^2-1},\\
\hat{p}_2 &= -\hat{p}_3 = \frac{2\pi\kappa\sqrt{x}\sqrt{x-\sin 2\gamma}}{x^2-1},\\
\tilde{p}_1 &= \frac{\pi\omega(2x-(1+x^2)\tan\gamma)}{x^2-1},\\
\tilde{p}_2 &= \frac{\pi\omega(2x-(1+x^2)\tan\gamma)}{x^2-1},\\
\tilde{p}_3 &= -\frac{\pi\omega(2x+(-3+x^2)\tan\gamma)}{x^2-1},\\
\tilde{p}_4 &= -\frac{\pi\omega(2x+(1-3x^2)\tan\gamma)}{x^2-1},
\end{aligned} \tag{2.35}$$

with

$$\sin 2\gamma = \frac{4\pi\sqrt{\lambda}JLM}{4\pi^2J^2+\lambda L^2M^2}, \quad \tan\gamma = \frac{\sqrt{\lambda}LM}{2\pi J}. \tag{2.36}$$

We find an expected square root cut $[\csc 2\gamma, \infty]$ connecting $\hat{p}_1$ and $\hat{p}_4$.

3 Semi-classical quantization of the BMN string

A powerful method for computing the semiclassical spectrum around string solutions is proposed in [44]. Here we begin with a review of this method for the reader's convenience.

The semiclassical spectrum around the BMN string is given by

$$\Delta = \Delta_{\rm cl} + \Delta_{1-\rm loop} + \delta\Delta, \tag{3.1}$$

where $\Delta_{\rm cl} + \Delta_{1-\rm loop}$ is the ground state energy, and $\delta\Delta$ is the energy of the excitations. To compute $\delta\Delta$, we add a perturbation $\delta p(x)$ to $p(x)$ associated with the classical solution. The perturbation $\delta p(x)$ has a single pole at x_n. The position x_n is determined by

$$p_i(x_n^{ij}) - p_j(x_n^{ij}) = 2\pi n. \tag{3.2}$$

The residues at the poles are

$$\mathrm{res}_{x=x_n^{ij}}\ \delta\hat{p}_k = (\delta_{ik}-\delta_{jk})\alpha(x_n^{ij})N_n^{ij}, \quad \mathrm{res}_{x=x_n^{ij}}\ \delta\tilde{p}_k = (\delta_{jk}-\delta_{ik})\alpha(x_n^{ij})N_n^{ij}, \tag{3.3}$$

where $i<j$ and N_n^{ij} is the excitation number for excitation with polarizations (ij) and mode number n and the function $\alpha(x)$ is defined as

$$\alpha(x) = \frac{4\pi}{\sqrt{\lambda}}\frac{x^2}{x^2-1}. \tag{3.4}$$

the residues at $x=\pm 1$ are synchronized as

$$\delta\{\hat{p}_1,\hat{p}_2,\hat{p}_3,\hat{p}_4|\tilde{p}_1,\tilde{p}_2,\tilde{p}_3,\tilde{p}_4\} = \frac{1}{x\pm 1}\delta\{\alpha_\pm,\alpha_\pm,\beta_\pm,\beta_\pm|\alpha_\pm,\alpha_\pm,\beta_\pm,\beta_\pm\} + ... \tag{3.5}$$

For each cut $\mathcal{C}^{ij}$ connecting $p_i(x)$ and $p_j(x)$ the perturbation $\delta p(x)$ satisfies

$$\delta p_i^+ - \delta p_j^- = 0, \quad x\in\mathcal{C}^{ij}. \tag{3.6}$$

where the superscript $\pm$ denotes above and below the cut.

The poles result in an energy shift

$$\delta\Delta = \sum_{n,ij} N_n^{ij} \Omega^{ij}(x_n^{ij}), \tag{3.7}$$

where Ω^{ij} are the off-shell fluctuation energies.

The asymptotic behaviour of the perturbation $\delta p(x)$ is

$$\delta\hat{p}_1 = O(x^{-1/2}), \quad \delta\hat{p}_4 = O(x^{-1/2}), \tag{3.8}$$

$$\begin{pmatrix} \delta\hat{p}_2 \\ \delta\hat{p}_3 \\ \hline \delta\tilde{p}_1 \\ \delta\tilde{p}_2 \\ \delta\tilde{p}_3 \\ \delta\tilde{p}_4 \end{pmatrix} = \frac{4\pi}{\sqrt{\lambda}x} \begin{pmatrix} +\frac{1}{2}\delta\Delta + N^{\hat{2}\hat{3}} + N^{\hat{2}\hat{4}} + N^{\hat{2}\tilde{3}} + N^{\hat{2}\tilde{4}} \\ -\frac{1}{2}\delta\Delta - N^{\hat{2}\hat{3}} - N^{\hat{1}\hat{3}} - N^{\tilde{1}\hat{3}} - N^{\tilde{2}\hat{3}} \\ \hline -N^{\tilde{1}\tilde{3}} - N^{\tilde{1}\tilde{4}} - N^{\tilde{1}\hat{3}} - N^{\tilde{1}\hat{4}} \\ -N^{\tilde{2}\tilde{3}} - N^{\tilde{2}\tilde{4}} - N^{\tilde{2}\hat{3}} - N^{\tilde{2}\hat{4}} \\ +N^{\tilde{1}\tilde{3}} + N^{\tilde{2}\tilde{3}} + N^{\hat{1}\tilde{3}} + N^{\hat{2}\tilde{3}} \\ +N^{\tilde{1}\tilde{4}} + N^{\tilde{2}\tilde{4}} + N^{\hat{1}\tilde{4}} + N^{\hat{2}\tilde{4}} \end{pmatrix} + \mathcal{O}(x^{-2}). \tag{3.9}$$

The order $O(x^{1/2})$ terms in the $\delta\hat{p}_1$ and $\delta\hat{p}_4$ are determined by the constraint (2.30). Since the classical solution already obey the constraint (2.30), one can show these filling fractions N_n^{ij} satisfy

$$\sum_{n,ij} n N_n^{ij} = 0. \tag{3.10}$$

using Riemann bilinear identity (see eqs. 3.38 and 3.44 in [11]).

We now determine the most general form of the perturbation $\delta p(x)$ for the BMN string. When small poles are added to $\delta\hat{p}_2(x)$ with a square root cut, the branch point will be slightly displaced so $\delta\hat{p}_2(x)$ behave like $\partial_{x_0}\sqrt{x-x_0} \sim 1/\sqrt{x-x_0}$ near the branch point $x_0 = \sin 2\gamma$. The most general form for $\delta\hat{p}_2(x)$ is $f(x)+g(x)/K(x)$ where f and g are rational functions of x and $K(x) = \sqrt{x}\sqrt{x-x_0}$. From (3.6) and inversion symmetry we get

$$\begin{pmatrix} \delta\hat{p}_1 \\ \delta\hat{p}_2 \\ \delta\hat{p}_3 \\ \delta\hat{p}_4 \end{pmatrix} = \begin{pmatrix} -f(1/x) - \frac{g(1/x)}{K(1/x)} \\ f(x) + \frac{g(x)}{K(x)} \\ f(x) - \frac{g(x)}{K(x)} \\ -f(1/x) + \frac{g(1/x)}{K(1/x)} \end{pmatrix}. \tag{3.11}$$

One can obtained all other off-shell fluctuation energies from the knowledge of $\Omega^{\tilde{2}\tilde{3}}$ and $\Omega^{\hat{2}\hat{3}}$ alone using the efficient method provided in [45]. From inversion symmetry we have

$$\Omega^{\tilde{1}\tilde{4}}(y) = -\Omega^{\tilde{2}\tilde{3}}(1/y) + \Omega^{\tilde{2}\tilde{3}}(0), \quad \Omega^{\tilde{1}\tilde{4}}(y) = -\Omega^{\tilde{2}\tilde{3}}(1/y) - 2. \tag{3.12}$$

the quasi-momenta for BMN string in $Sch_5 \times S^5$ are pairwise symmetric up to constant terms:

$$\hat{p}_1 = -\hat{p}_4, \quad \hat{p}_2 = -\hat{p}_3, \tag{3.13}$$

$$\tilde{p}_1 = -\tilde{p}_4 + 2\pi\kappa\sin\gamma, \quad \tilde{p}_2 = -\tilde{p}_3 - 2\pi\kappa\sin\gamma. \tag{3.14}$$

So the off-shell fluctuation energies satisfies

$$\Omega^{ij}(y) = \frac{1}{2}(\Omega^{ii'}(y) + \Omega^{j'j}(y)), \tag{3.15}$$

where

$$(\hat{1}, \hat{2}, \tilde{1}, \tilde{2}, \hat{3}, \hat{4}, \tilde{3}, \tilde{4}) = (\hat{4}, \hat{3}, \tilde{4}, \tilde{3}, \hat{2}, \hat{1}, \tilde{2}, \tilde{1}). \tag{3.16}$$

3.1 Fluctuation energies of excitations

We first consider the excitation $\hat{2}\hat{3}$. We have $\delta\tilde{p}_i = 0$, for $i = 1, 2, 3, 4$, and therefore $f(x) = 0$. We take the following ansatz

$$g(x) = \sum_n \frac{\alpha(x_n^{\hat{2}\hat{3}})K(x_n^{\hat{2}\hat{3}})N_n^{\hat{2}\hat{3}}}{x - x_n^{\hat{2}\hat{3}}} + \frac{2\pi}{\sqrt{\lambda}}\delta\Delta + \frac{4\pi}{\sqrt{\lambda}}\sum_n N_n^{\hat{2}\hat{3}}. \tag{3.17}$$

Large x asymptotic of $\delta\hat{p}_1$ gives

$$\delta\Delta = \sum_n N_n^{\hat{2}\hat{3}}\Omega^{\hat{2}\hat{3}}(x_n^{\hat{2}\hat{3}}), \tag{3.18}$$

$$\Omega^{\hat{2}\hat{3}}(x) = \frac{2 - 2x^2 + 2xK(x)}{x^2 - 1}, \tag{3.19}$$

and the level matching condition

$$\sum_n nN_n^{\hat{2}\hat{3}} = 0. \tag{3.20}$$

We next consider the S^3 excitation $\tilde{2}\tilde{3}$. We start with the following ansatz

$$g(x) = \frac{\alpha_1 K(1)}{x - 1} + \frac{\alpha_2 K(-1)}{x + 1} + \frac{2\pi}{\sqrt{\lambda}}\delta\Delta, \tag{3.21}$$

$$\delta\tilde{p}_2 = -\delta\tilde{p}_3 = \frac{\alpha_1}{x - 1} + \frac{\alpha_2}{x + 1} - \sum_n \frac{\alpha(x_n^{\tilde{2}\tilde{3}})N_n^{\tilde{2}\tilde{3}}}{x - x_n^{\tilde{2}\tilde{3}}}. \tag{3.22}$$

Large x asymptotic of $\delta\tilde{p}_1$ and $\delta\tilde{p}_4$ give

$$\alpha_1 + \alpha_2 = \sum_n \alpha(x_n^{\tilde{2}\tilde{3}})N_n^{\tilde{2}\tilde{3}}\frac{1}{(x_n^{\tilde{2}\tilde{3}})^2}, \tag{3.23}$$

$$\alpha_1 - \alpha_2 = \sum_n \alpha(x_n^{\tilde{2}\tilde{3}})N_n^{\tilde{2}\tilde{3}}\frac{1}{x_n^{\tilde{2}\tilde{3}}}. \tag{3.24}$$

Large x asymptotic of $\delta\hat{p}_1$ give

$$-\alpha_1 K(1) + \alpha_2 K(-1) + \frac{2\pi}{\sqrt{\lambda}}\delta\Delta = -\alpha_1(\cos\gamma - \sin\gamma) - \alpha_2(\cos\gamma + \sin\gamma) + \frac{2\pi}{\sqrt{\lambda}}\delta\Delta = 0. \tag{3.25}$$

The filling fractions satisfy

$$\sum_n nN_n^{\tilde{2}\tilde{3}} = 0, \tag{3.26}$$

we can solve these equations and obtain

$$\delta\Delta = \sum_n N_n^{\tilde{2}\tilde{3}}\Omega^{\tilde{2}\tilde{3}}(x_n^{\tilde{2}\tilde{3}}), \tag{3.27}$$

where

$$\Omega^{\tilde{2}\tilde{3}}(x) = \frac{2\cos\gamma - 2x\sin\gamma}{x^2 - 1}. \tag{3.28}$$

Finally using (3.12) and (3.15) we find all the off-shell fluctuation energies

$$
\begin{aligned}
\Omega^{\hat{2}\hat{3}}(x) &= \frac{2-2x^2+2x\sqrt{x}\sqrt{x-\sin 2\gamma}}{x^2-1}, \\
\Omega^{\hat{1}\hat{4}}(x) &= \frac{2\sqrt{1-x\sin 2\gamma}}{x^2-1}, \\
\Omega^{\hat{1}\hat{3}}(x) = \Omega^{\hat{2}\hat{4}}(x) &= \frac{1-x^2+x\sqrt{x}\sqrt{x-\sin 2\gamma}+\sqrt{1-x\sin 2\gamma}}{x^2-1}, \\
\Omega^{\tilde{2}\tilde{3}}(x) = \Omega^{\tilde{1}\tilde{4}}(x) = \Omega^{\tilde{1}\tilde{3}}(x) = \Omega^{\tilde{2}\tilde{4}}(x) &= \frac{2\cos\gamma - 2x\sin\gamma}{x^2-1}, \\
\Omega^{\hat{2}\tilde{3}}(x) = \Omega^{\tilde{2}\hat{3}}(x) = \Omega^{\hat{2}\tilde{4}}(x) = \Omega^{\tilde{2}\hat{4}}(x) &= \frac{1+\cos\gamma - x\sin\gamma - x^2 + x\sqrt{x}\sqrt{x-\sin 2\gamma}}{x^2-1}, \\
\Omega^{\hat{1}\tilde{4}}(x) = \Omega^{\tilde{1}\hat{4}}(x) = \Omega^{\hat{1}\tilde{3}}(x) = \Omega^{\tilde{1}\hat{3}}(x) &= \frac{\cos\gamma - x\sin\gamma + \sqrt{1-x\sin 2\gamma}}{x^2-1}.
\end{aligned}
\tag{3.29}
$$

We now solve the pole position x_n. We choose the solution $|x_n| > 1$ for small LM to be physical poles. We have

$$
\begin{aligned}
& x_n^{\hat{2}\hat{3}} = \frac{2}{n\sqrt{\lambda}} J + O(1), \quad x_n^{\hat{1}\hat{4}} = \frac{2\sqrt{L^2M^2\pi^{-2}+n^2} - 2LM\pi^{-1}}{\sqrt{\lambda}n^2} J + O(1), \\
& x_n^{\hat{1}\hat{3}} = x_n^{\hat{2}\hat{4}} = \frac{(2\pi n - LM)}{\pi\sqrt{\lambda}n^2} J + O(1), \\
& x_n^{\tilde{2}\tilde{3}} = x_n^{\tilde{1}\tilde{3}} = \frac{\sqrt{\pi J^2 + \lambda n(\pi n - LM)} + \sqrt{\pi} J}{\sqrt{\pi}\sqrt{\lambda} n}, \quad x_n^{\tilde{1}\tilde{4}} = x_n^{\tilde{2}\tilde{4}} = \frac{\sqrt{\pi}\sqrt{\pi J^2 + \lambda n(LM+\pi n)} + \pi J}{\sqrt{\lambda}(LM+\pi n)}, \\
& x_n^{\tilde{2}\hat{3}} = x_n^{\tilde{1}\hat{3}} = \frac{\sqrt{16\pi^2 J^2 + \lambda(LM-4\pi n)^2} + 4\pi J}{\sqrt{\lambda}(LM+4\pi n)}, \\
& x_n^{\hat{2}\tilde{3}} = \frac{\sqrt{16\pi^2 J^2 + \lambda(3LM-4\pi n)^2} + 4\pi J}{\sqrt{\lambda}(4\pi n - LM)}, \quad x_n^{\hat{2}\tilde{4}} = \frac{\sqrt{16\pi^2 J^2 + \lambda(LM+4\pi n)^2} + 4\pi J}{\sqrt{\lambda}(3LM+4\pi n)}, \\
& x_n^{\hat{1}\tilde{4}} = \frac{(LM+4\pi n)\left(\sqrt{16\pi^2 J^2 + \lambda(3LM+4\pi n)^2} + 4\pi J\right)}{\sqrt{\lambda}(3LM+4\pi n)^2}, \\
& x_n^{\hat{1}\tilde{3}} = \frac{(4\pi n - 3LM)\left(\sqrt{16\pi^2 J^2 + \lambda(LM-4\pi n)^2} + 4\pi J\right)}{\sqrt{\lambda}(LM-4\pi n)^2}, \\
& x_n^{\tilde{2}\hat{4}} = x_n^{\tilde{1}\hat{4}} = \frac{(4\pi n - LM)\left(\sqrt{16\pi^2 J^2 + \lambda(LM+4\pi n)^2} + 4\pi J\right)}{\sqrt{\lambda}(LM+4\pi n)^2}.
\end{aligned}
\tag{3.30}
$$

The exact expressions of $x_n^{\hat{i}\hat{j}}$ are very complex, so we only consider the leading order terms in the large J expansion. When $LM \neq 0$, a finite number of $x_n^{\hat{1}j}$ ($x_n^{i\hat{4}}$) will enter the cut connecting $\hat{p}_1$ and $\hat{p}_4$ and become $x_n^{\hat{4}j}$ ($x_n^{i\hat{1}}$).

Pluging x_n into the off-shell fluctuation energies, in the large J limit we get the on-shell fluctuation

frequencies

$$
\begin{aligned}
\Omega^{\hat{2}\hat{3}}(x_n^{\hat{2}\hat{3}}) &= \frac{\lambda(\pi n^2 - LMn)}{2\pi J^2} + O\left(J^{-3}\right), \\
\Omega^{\hat{1}\hat{4}}(x_n^{\hat{1}\hat{4}}) &= \frac{\lambda}{2J^2}\left(\sqrt{\frac{L^2M^2n^2}{\pi^2} + n^4} + \frac{LMn}{\pi}\right) + O\left(J^{-3}\right), \\
\Omega^{\hat{1}\hat{3}}(x_n^{\hat{1}\hat{3}}) &= \Omega^{\hat{2}\hat{4}}(x_n^{\hat{2}\hat{4}}) = \frac{\lambda n^2}{2J^2} + O\left(J^{-3}\right), \\
\Omega^{\tilde{2}\tilde{3}}(x_n^{\tilde{2}\tilde{3}}) &= \Omega^{\tilde{1}\tilde{3}}(x_n^{\tilde{1}\tilde{3}}) = \frac{\lambda(\pi n^2 - LMn)}{2\pi J^2} + O\left(J^{-3}\right), \\
\Omega^{\tilde{2}\tilde{4}}(x_n^{\tilde{2}\tilde{4}}) &= \Omega^{\tilde{1}\tilde{4}}(x_n^{\tilde{1}\tilde{4}}) = \frac{\lambda(\pi n^2 + LMn)}{2\pi J^2} + O\left(J^{-3}\right), \\
\Omega^{\tilde{2}\hat{3}}(x_n^{\tilde{2}\hat{3}}) &= \Omega^{\tilde{1}\hat{3}}(x_n^{\tilde{1}\hat{3}}) = \frac{\lambda\left(16\pi^2n^2 - 8\pi LMn - 3L^2M^2\right)}{32\pi^2J^2} + O\left(J^{-3}\right), \qquad (3.31)\\
\Omega^{\hat{2}\tilde{3}}(x_n^{\hat{2}\tilde{3}}) &= \frac{\lambda\left(16\pi^2n^2 - 24\pi LMn + 5L^2M^2\right)}{32\pi^2J^2} + O\left(J^{-3}\right), \\
\Omega^{\hat{2}\tilde{4}}(x_n^{\hat{2}\tilde{4}}) &= \frac{\lambda\left(16\pi^2n^2 + 8\pi LMn - 3L^2M^2\right)}{32\pi^2J^2} + O\left(J^{-3}\right), \\
\Omega^{\hat{1}\tilde{4}}(x_n^{\hat{1}\tilde{4}}) &= \frac{\lambda(4\pi n + 3LM)^2}{32\pi^2J^2} + O\left(J^{-3}\right), \\
\Omega^{\hat{1}\tilde{3}}(x_n^{\hat{1}\tilde{3}}) &= \frac{\lambda(4\pi n - LM)^2}{32\pi^2J^2} + O\left(J^{-3}\right), \\
\Omega^{\tilde{2}\hat{4}}(x_n^{\tilde{2}\hat{4}}) &= \Omega^{\tilde{1}\hat{4}}(x_n^{\tilde{1}\hat{4}}) = \frac{\lambda(4\pi n + LM)^2}{32\pi^2J^2} + O\left(J^{-3}\right).
\end{aligned}
$$

Then we obtained the energy shift $\delta\Delta$ given by (3.7).

3.2 One-loop shift

The one loop shift is equal to one half of the graded sum of all fluctuation mode frequencies. Using zeta function regularization, we have

$$\sum_{n\in\mathbb{Z}}\left((n+q)^2 + pn\right) = q^2 + \zeta(-2, 1+q) + \zeta(-2, 1-q) = 0. \tag{3.32}$$

Therefore when we compute the one loop shift energy at order J^{-2}, only the contribution from $\Omega^{\hat{1}\hat{4}}$ is nontrivial. Then we sum over the energies of the $sl(2)$ modes to get the one-loop shift:

$$
\begin{aligned}
\Delta_{1-\mathrm{loop}} &= \frac{1}{2}\sum_{n\in\mathbb{Z}}\Omega^{\hat{1}\hat{4}}(x_n^{\hat{1}\hat{4}}) \\
&= \sum_{n=1}^{\infty}\left(\sum_{k=0}^{\infty}\binom{1/2}{k}n^{2-2k}\pi^{-2k}L^{2k}M^{2k}\right)\frac{\lambda}{2J^2} + O\left(J^{-3}\right) \\
&= \left(\sum_{k=0}^{\infty}\binom{1/2}{k}\zeta(2k-2)\pi^{-2k}L^{2k}M^{2k}\right)\frac{\lambda}{2J^2} + O\left(J^{-3}\right) \\
&= \left(-\frac{\lambda L^2M^2}{8\pi^2} - \frac{\lambda L^4M^4}{96\pi^2} + \frac{\lambda L^6M^6}{2880\pi^2} + O\left(L^8M^8\right)\right)\frac{1}{J^2} + O\left(J^{-3}\right). \qquad (3.33)
\end{aligned}
$$

4 Comparison between the string and the gauge theory results

Comparison between the results obtained in the gauge theory and string theory is possible in the large spin regime with $J\to\infty$ and λ/J^2 kept fixed and small (see e.g. [26, 46–48]). Type IIB superstring

in $Sch_5 \times S_5$ is dual to null dipole deformed $\mathcal{N}=4$ super Yang-Mills. The $sl(2)$ sector nontrivially affected by the deformation has been studied in [23]. The one-loop spectrum of the $sl(2)$ sector can be obtained by Baxter equation. It is proposed in [23] that the Baxter equation takes the same form as in the undeformed case

$$t(u)Q(u)=(u+i/2)^J Q(u+i)+(u-i/2)^J Q(u-i), \tag{4.1}$$

and

$$t(u)=2u^J+LMJu^{J-1}+... \tag{4.2}$$

The 1-loop energy is given by

$$\Delta^{(1)}=\frac{i\lambda}{8\pi^2}\partial_u \log \frac{Q(u+i/2)}{Q(u-i/2)}\bigg|_{u=0}. \tag{4.3}$$

We now solve the Baxter equation in the expansion in LM. At each order in LM, the Q-function is simply a polynomial. We write the ansatz

$$Q(u)=\sum_{k=0}^{\infty} p_{k+m}(u)L^k M^k, \tag{4.4}$$

where p_{k+m} is a polynomial in u of degree $k+m$. The small LM expansion of Q can be interpreted as a function with a finite number of zeros near the Bethe roots in the undeformed limit and an infinite number of zeros of order $L^{-1}M^{-1}$. In the string picture the zeros of order $L^{-1}M^{-1}$ correspond to the cut connecting $\hat{p}_1$ and $\hat{p}_4$.

Substituting the above ansatz into the Baxter equation, we can determine $Q(u)$ up to multiplication by a function in LM. We consider the following two solutions:

$$Q_0=1-LMu+L^2M^2\frac{Ju^2}{2(J+1)}+O\left(L^3M^3\right), \tag{4.5}$$

$$\begin{aligned}Q_1=&(u-u_n)+\frac{LM\left(-2Ju^2+2Ju_n^2+4u_n^2+1\right)}{2(J+2)}\\&+L^2M^2\frac{1}{2(J+2)^3(J+3)}(J^4(u^3+u_nu^2-2u_n^3)+2J^3(2u^3+3u_nu^2-9u_n^3-u_n)\\&+J^2\left(4u^3+12u_nu^2-60u_n^3-11u_n\right)-22J\left(4u_n^3+u_n\right)-12\left(4u_n^3+u_n\right))+O\left(L^3M^3\right), \qquad (4.6)\end{aligned}$$

$$\tag{4.7}$$

with

$$u_n=\frac{1}{2}\cot\left(\frac{\pi n}{J}\right). \tag{4.8}$$

In the undeformed case, Q_0 and Q_1 correspond to the ground state and one particle state respectively. Although not shown here, we have computed Q_0 and Q_1 to L^4M^4 order. Then the energies of the

ground state and one particle state are

$$\Delta_0^{(1)} =\frac{\lambda L^2M^2}{8\pi^2(J+1)} - \frac{\lambda L^4M^4}{96\pi^2(J+1)^2} + O\left(L^5M^5\right), \tag{4.9}$$

$$\begin{aligned}\Delta_1^{(1)} =&\frac{\lambda}{2\pi^2\left(4u_n^2+1\right)} + \frac{2\lambda LMu_n}{\pi^2(J+2)\left(4u_n^2+1\right)}\\ &+ \frac{\lambda L^2M^2\left(4(J(J(J+10)+20)+24)u_n^2+(J-6)(J+2)^2\right)}{8\pi^2(J+2)^3(J+3)\left(4u_n^2+1\right)}\\ &- \lambda L^3M^3\frac{48J^3u_n^3+4(J+2)^2(J(4J+7)+12)u_n}{3\pi^2(J+2)^5(J+3)(J+4)\left(4u_n^2+1\right)}\\ &- \lambda L^4M^4\frac{c_4u_n^4+c_2u_n^2+c_0}{96\pi^2(J+2)^7(J+3)^3(J+4)(J+5)\left(4u_n^2+1\right)} + O\left(L^5M^5\right), \end{aligned}\tag{4.10}$$

$$\begin{aligned} c_4 =&96J^3\left(J^5+36J^4-92J^3-1808J^2-4656J-2880\right),\\ c_2 =&4(J+2)^2(J^8+18J^7+495J^6+2474J^5+4520J^4+12304J^3\\ &+35568J^2+33696J+17280),\\ c_0 =&(J+2)^4(J+4)\left(J^5+6J^4-211J^3-738J^2-756J-1080\right). \end{aligned}$$

In large J limit, we get

$$\Delta_0^{(1)} = \frac{\lambda L^2M^2}{8\pi^2J} + \left(-\frac{\lambda L^2M^2}{8\pi^2J^2} - \frac{\lambda L^4M^4}{96\pi^2J^2} + O\left(L^5M^5\right)\right) + O(J^{-3}), \tag{4.11}$$

$$\Delta_1^{(1)} = \frac{\lambda L^2M^2}{8\pi^2J} + \left(\frac{\lambda n^2}{2J^2} + \frac{\lambda LMn}{\pi J^2} + \frac{\lambda L^2M^2}{8\pi^2J^2} - \frac{\lambda L^4M^4\left(\pi^2n^2+6\right)}{96J^2\left(\pi^4n^2\right)} + O\left(L^5M^5\right)\right) + O(J^{-3}). \tag{4.12}$$

Assuming that the interaction between particles can be neglected in the large J limit as in the undeformed case, the energy of an excited state above ground state energy is

$$\Delta^{(1)} - \Delta_0^{(1)} = \sum_n N_n\left(\frac{\lambda n^2}{2J^2} + \frac{\lambda L^2M^2}{4\pi^2J^2} - \frac{\lambda L^4M^4}{16\left(\pi^4J^2n^2\right)} + O\left(L^5M^5\right)\right) + O(J^{-3}), \tag{4.13}$$

where N_n is excitation number for mode number n, and we assume that the total momentum is zero.

To compare the spectral curve result with the spin chain result, we expand $\Omega^{\hat{1}\hat{4}}(x_n^{\hat{1}\hat{4}})$ for small LM and obtain the energy shift of the $sl(2)$ sector

$$\delta\Delta = \sum_n N_n^{\hat{1}\hat{4}}\Omega^{\hat{1}\hat{4}}(x_n^{\hat{1}\hat{4}}) = \sum_n N_n^{\hat{1}\hat{4}}\left(\frac{\lambda n^2}{2J^2} + \frac{\lambda L^2M^2}{4\pi^2J^2} - \frac{\lambda L^4M^4}{16\left(\pi^4J^2n^2\right)} + O\left(L^5M^5\right)\right) + O(J^{-3}). \tag{4.14}$$

where we have used the level matching condition. The result agrees with (4.13).

The energy of the spin chain ground state to order L^6M^6 has been computed in [23]

$$\begin{aligned}\Delta_0^{(1)} &= \frac{\lambda}{4\pi^2}\left(\frac{L^2M^2}{2(J+1)} - \frac{L^4M^4}{24(J+1)^2} + \frac{L^6M^6(J^2+J+2)}{720(J+1)^3(J+2)} + O\left(L^8M^8\right)\right),\\ &= \frac{\lambda L^2M^2}{8\pi^2J} + \left(-\frac{\lambda L^2M^2}{8\pi^2} - \frac{\lambda L^4M^4}{96\pi^2} + \frac{\lambda L^6M^6}{2880\pi^2} + O\left(L^8M^8\right)\right)\frac{1}{J^2} + O\left(J^{-3}\right),\end{aligned}\tag{4.15}$$

The order J^{-1} term matches the classical quantity $\Delta_{\rm cl} - J$, and the order J^{-2} terms perfectly match the one-loop shift $\Delta_{1-\rm loop}$ given in (3.33).

5 Conclusion and discussion

In this paper we study the algebraic curve for superstring in $Sch_5 \times S^5$ and its application to the spectral problem. The asymptotic properties of the quasi-momenta for strings in $Sch_5 \times S^5$ are nontrivial. The point at infinity is a branch point of a cut connecting two Riemann sheets. We compute the semiclassical spectrum of the BMN string. Remarkably, we show that in the large J limit the string results match the gauge field results obtained by Baxter equation. We provide a detailed test of the Schrödinger/dipole CFT duality.

Our results encourage further exploration of integrability in Schrödinger/dipole CFT duality. It would be nice to derive the full quantum spectral curve of null dipole deformed $\mathcal{N} = 4$ super Yang-Mills theory, because the quasi-momenta are related to the quantum spectral curve in the strong coupling limit. It is also worth trying to obtain higher-order corrections on the field theory side to get a precise match with string theory predictions. One can also study the three dimensional counterpart of Sch_5, the warped AdS_3. We hope that integrability would be a powerful tool for the spectral problem of warped AdS_3/dipole CFT duality [49].

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